SMOOTH MANIFOLDS FALL 2022 - MIDTERM REVIEW

1. Definitions

Definition 1 (Topological Manifold)**.** A topological space M is *locally (n-)Euclidean* if for every $x \in M$, there exists a neigbhorhood $U \subset M$, an open set $V \subset \mathbb{R}^n$ and a homeomorphism $\varphi : U \to V$. The pair (U, φ) is called a *chart* of M (φ is often called a chart as well). A *topological manifold* is a topolgoical space which is Hausdorff, second countable and locally Euclidean.

Definition 2 (Smooth Manifold)**.** Let M be a topological manifold. A set $\mathcal{A}$ of charts is called an *atlas* if $\bigcup_{(U,\varphi)\in\mathcal{A}} U = M$ (ie, the domains of the charts cover M). The atlas is called *smooth* if whenever $(U, \varphi), (V, \psi) \in \mathcal{A}$ are charts such that $U \cap V \neq \emptyset$, $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \mathbb{R}^n$ is a diffeomorphism onto its image. A maximal smooth atlas is often called a *smooth structure*. A smooth manifold is a topological manifold with a distinguished smooth structure.

Definition 3 (Tangent Bundle)**.** Let M be a smooth manifold, and $p \in M$. The *tangent space at* p is an n-dimensional vector space of equivalence classes. If (U, φ) is a chart such that $p \in U$, then we define $T_p^{\varphi}M$ with the vector space $T_{\varphi(p)}\mathbb{R}^n = \mathbb{R}^n$. We say that $v \in T_p^{\varphi}M$ and $w \in T_p^{\psi}M$ are equivalent if $D(\psi \circ \varphi^{-1})(v) = w$. T_pM is the vector space of equivalence classes in $\bigcup\limits_{(U,\varphi)} T_p^{\varphi}M$.

Definition 4 (Derivative conditions)**.** If M and N are manifolds, and $F : M \to N$ is a C^∞ map, we say that F is

- a *submersion* if $DF(p)$ is surjective for every $p \in M$,
- a *immersion* if $DF(p)$ is injective for every $p \in M$,
- a *emedding* if it is an immersion and a homeomorphism onto its image, and
- a *local diffeomorphism* if $DF(p)$ is an isomorphism for every $p \in M$.

Definition 5 (Vector Bundle)**.** Let E and B be smooth manifolds, and $\dim(B) = n$. We say that E is a *vector bundle* over B if there exists a submersion $\pi : E \to B$ and an atlas $\mathcal{A}$ of charts $\hat{\varphi} : \hat{U} \to \mathbb{R}^n \times \mathbb{R}^m$, where $\hat{U} = \pi^{-1}(U)$ for some open set $U \subset B$. We require that the atlas satisfies that if $(\hat{U}, \hat{\varphi}) \in \mathcal{A}$, there exists a chart φ defined on U such that $\varphi \circ \pi = p \circ \hat{\varphi}$, where p is the linear projection from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$, and $\hat{\varphi}(\hat{U}) = \varphi(U) \times \mathbb{R}^m$. Furthermore if $\hat{\varphi}, \hat{\psi} \in \mathcal{A}$, then $\hat{\varphi} \circ \hat{\psi}^{-1}|_{\{x\}\times\mathbb{R}^m}$ is a linear isomorphism.

Definition 6 (Sections of vector bundles)**.** If $\pi : E \to B$ is a vector bundle, and *section* is a map $\sigma : B \to E$ such that $\pi \circ \sigma = \mathrm{Id}_B$. The section is called smooth if σ is C^∞. A section of TM is called a *vector field*.

Definition 7 (Local and Global Derivations)**.** If M is a C^∞ manifold and $p \in M$, the *germ of a* C^∞ *function* at p is an equivalence class of functions under the equivalence relation $f \sim g$ if and only if there exists $U \subset M$ open and containing p such that $f|_U = g|_U$. Let $C_p^\infty(M)$ denote the set equivalence classes of real-valued functions at p. A *(local) derivation (at p)* is a map $\delta : C_p^\infty(M) \to \mathbb{R}$ such that if $f, g \in C_p^\infty(M)$, then

$$\delta(f \cdot g) = f(p)\delta(g) + g(p)\delta(g). \tag{1.1}$$

A *global derivation* on M is a map $\delta : C^\infty(M) \to C^\infty(M)$ satisfying the following generalization of (1.1):

$$\delta(f \cdot g)(p) = f(p) \cdot \delta(g)(p) + g(p) \cdot \delta(f)(p).$$

Each tangent vector induces a local derivation at its basepoint, and each vector field induces a global derivation, which we denote by $\delta_X(f) = X \cdot f$.

Definition 8. An *ℓ-distribution* or *ℓ-dimensional subbundle* on M is an assignment to each $x \in M$ an ℓ-dimensional subspace $E(x) \subset T_xM$. The distribution is smooth if for every $p \in M$, there exists an open set $U \subset M$ containing p and vector fields $X_1, \dots, X_\ell$ which do not vanish on U such that $E(x) = \operatorname{span}_{\mathbb{R}} \{X_1(x), \dots, X_\ell(x)\}$ for every $x \in U$. We say that E is integrable if it is the tangent bundle of some foliation $\mathcal{F}$ (see the subsequent definition).

Definition 9. An *ℓ-foliation* on M is a smooth atlas $\mathcal{F}$ such that for every $\varphi, \psi \in \mathcal{F}$, $\varphi \circ \psi^{-1}(\mathbb{R}^\ell \times \{y\}) \subset \mathbb{R}^\ell \times \left\{\varphi(\psi^{-1}(y))\right\}$ for every $y \in \mathbb{R}^{n-\ell}$. The *tangent bundle to* $\mathcal{F}$ is the distribution $T\mathcal{F}(p) = D\varphi^{-1}(\varphi(p))(\mathbb{R}^\ell)$, where $\varphi \in \mathcal{F}$ is any chart containing p in its domain. If E is a distribution such that $E = T\mathcal{F}$ for some foliation $\mathcal{F}$, we say that E is *integrable*.

Definition 10 (Flow)**.** A *flow* on a manifold M is a C^∞ map $F : \mathbb{R} \times M \to M$ satisfying

$$F(t+s, x) = F(t, F(s, x)) \tag{1.2}$$

for every $t, s \in \mathbb{R}$, $x \in M$ and $F(0, x) = x$ for every $x \in X$. A *local flow* is a map F defined only on an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$ satisfying equation (1.2) whenever it makes sense. Flows are often denoted as $F_t(x) := F(t, x)$, so that (1.2) becomes $F_t \circ F_s = F_{t+s}$.

Definition 11 (Lie bracket)**.** Let X, Y be vector fields on M. The *Lie derivative of Y along X* is the vector field

$$[X, Y](p) = \left.\frac{d}{dt}\right|_{t=0} (F_t^Y)_*(X)$$

where F_t^Y is the (local) flow generated by Y. The Lie bracket of X and Y acts on C^∞ functions via the formula

$$[X, Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f).$$

If E is a distribution on M, then a vector field X is called *subordinate to* E if and only if for every $p \in M$, $X(p) \in E(p)$. We say that E is *involutive* if for any two vector fields X and Y subordinate to E, $[X, Y]$ is subordinate to E.

Definition 12 (Transversality)**.** If M, N are smooth manifolds, $Q \subset N$ is an embedded submanifold, and $f : M \to N$ is a C^∞ map, then f is called *transverse to* Q if for every $p \in M$ such that $f(p) \in Q$, $\operatorname{Im}(Df(p)) + T_{f(p)}Q = T_{f(p)}N$.

2. Theorems

Theorem 13 (Submerison Theorem)**.** *If $F : M \to N$ is a submersion, then preimages $\left\{F^{-1}(n) : n \in N\right\}$ are the leaves of a foliation of M. Furthermore, every leaf is embedded.*

Theorem 14 (Frobenius Theorem)**.** *A distribution E on a manifold M is involutive if and only if it is integrable.*

Theorem 15. *If X is a C^∞ vector field on a smooth manifold M, there exists a unique (local) C^∞ flow $\varphi_t : M \to M$ such that $X(p) = \left.\frac{d}{dt}\right|_{t=0} \varphi_t(p)$.*

Theorem 16. *The Lie bracket is well defined, and the definitions via flows and derivations are equivalent. The Lie bracket satisfies the following:*

- *(Product rule)* $[X, fY] = (X \cdot f)Y + f[X, Y]$
- *(Anticommutativity)* $[X, Y] = -[Y, X]$
- *(Bilinearity)* $[X, Y + Z] = [X, Y] + [X, Z]$
- *(Jacobi Identity)* $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

Theorem 17 (Transversality Theorem)**.** *If M, N are smooth manifolds, $Q \subset N$ is an embedded submanfiold, and $f : M \to N$ is transverse to Q, then $\hat{Q} = f^{-1}(Q)$ is an embedded submanifold of M, such that $\operatorname{codim}(\hat{Q}) = \operatorname{codim}(Q)$. In the case when f is an embedding, the intersection $f(M) \cap Q$ is an embedded submanifold of N diffeomorphic to $\hat{Q}$.*

Theorem 18. *The following properties are open properties on compact manifolds (ie, stable under perturbations): submersion, immersion, embedding, local diffeomorphism, diffeomorphism, and transversality to a fixed submanifold $Q \subset N$.*